THE EXISTENCE OF PERIODIC WAVES WHICH DEGENERATE INTO A SOLITARY WAVE

(SUSHCHESTVOVANIE PERIODICHESKIKH VOLN, VYROZHDAIUSHCHIKHSIA V UEDINENNUIU)

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A. M. TER-KRIKOROV
(Moscow)

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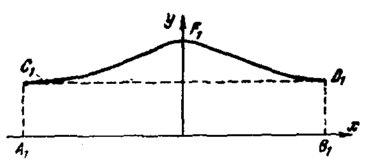
Korteweg and De Vries [1] were the first to make an approximate investigation, in 1895, of long surface water waves degenerating into a solitary wave when the wavelength tends to infinity. The equation of the profile of these waves is expressed by a Jacobi elliptical function in *cnx*. These authors dubbed these waves "cnoidal" because of the *cn* sound.

Many studies, both theoretical and experimental, have been carried out on cnoidal waves in recent years.

Lavrent'evym [2] in 1946 gave a formal proof of the existence of the solitary wave based on variational principles in conformal representation. In 1954 Friedrichs and Hyers [3] put forward a simpler proof which was based on the general theorems of functional analysis. Littman [4] demonstrated the existence of a certain class of cnoidal wave. This class does not include those degenerating into a solitary wave when the wavelength tends to infinity. Here we give a proof of existence which is valid over the whole range of cnoidal waves.

1. Definition of problem. We deal with a steady periodic wave, of length $2L_1$, moving at constant velocity c in a channel with a smooth horizontal bottom surface and filled with an ideal incompressible liquid. It is assumed that the wave is symmetrical about a vertical axis passing through the peak. It is a well-known fact that the velocity of a wave moving over a smooth horizontal bottom surface is an indeterminate quantity. One can define the wave velocity c, for instance, as the mean velocity of particles over the bottom [5]



$$c = \frac{1}{L_1}\int_0^{L_1} v(s)\,ds. \qquad (1.1)$$

Let us take a system of Cartesian coordinates which are tied to the wave as shown in the accompanying figure. The motion, with respect to these coordinates, will be a steady state. Put $z_1 = x_1 + iy_1$. It is assumed that there is no turbulence. The velocity potential $\phi_1(x_1, y_1)$ and stream function $\psi_1(x_1, y_1)$ will be conjugate harmonic functions, $w_1(z_1) = \phi_1(x_1, y_1) + i\psi_1(x_1, y_1)$ will be a function which is analytic in the curved quadrilateral $A_1B_1C_1D_1$ (Fig.). At the free boundary the following condition of constant pressure should hold:

$$\frac{1}{2}\left|\frac{dw_1}{dz_1}\right|^2 + 2gY_1(x_1) = \text{const} \tag{1.2}$$

Here g is the acceleration of gravity, $y_1 = Y_1(x_1)$ is the equation of the free boundary. Because the motion is a steady state the free boundary and the bottom should be streamlines

$$\psi_1 = 0 \quad \text{for } y_1 = 0, \qquad \psi_1 = Q \quad \text{for } y_1 = Y_1(x_1) \tag{1.3}$$

where Q is the discharge of fluid through a channel cross section. In view of symmetry about lines A_1C_1 and B_1D_1 the velocities are horizontal, i.e. $\partial\phi/\partial s = 0$ along A_1C_1 and B_1D_1. Therefore

$$\varphi_1(L_1, y_1) = -\varphi_1(-L_1, y_1) = d \tag{1.4}$$

where d is some constant which can easily be expressed in terms of L_1 and c. In actual fact it follows from (1.1) that $cL_1 = d$.

We introduce non-dimensional variables

$$z = z_1\frac{c}{Q}, \quad w = \frac{w_1}{Q}, \quad Y(x) = Y_1(x_1)\frac{c}{Q}, \quad L_1 = \frac{\pi O}{c\lambda}, \quad d = \frac{Q\pi}{\lambda} \tag{1.5}$$

It is obvious from this that the condition $cL_1 = d$ will be satisfied. We arrive at the following mathematical problem, namely, to find a function $Y(x)$ and a function $w(z)$ continuous over the interval $(-\pi/\lambda, \pi/\lambda)$ which are analytic in region $ABCD$ and which satisfy the boundary conditions

$$\frac{1}{2}\left|\frac{dw}{dz}\right|^2 + \nu Y(x) = \text{const} \quad \text{for } y = Y(x) \quad \left(\nu = \frac{gQ}{c^3}\right) \tag{1.6}$$

$$\psi = 0 \quad \text{for } y = 0, \qquad \psi = 1 \quad \text{for } y = Y(x) \tag{1.7}$$

$$\varphi = \frac{\pi}{\lambda} \quad \text{for } x = \frac{\pi}{\lambda}, \qquad \varphi = -\frac{\pi}{\lambda} \quad \text{for } x = -\frac{\pi}{\lambda} \tag{1.8}$$

Condition (1.8) is equivalent to the requirement for periodicity in a solution. Now let us change the variable

$$\chi(w) = \theta + i\tau = i \ln \left(\nu^{-\frac{1}{3}} \frac{dw}{dz}\right) \tag{1.9}$$

In the complex potential plane, rectangle $0 < \psi < 1$, $-\pi/\lambda < \phi < \pi/\lambda$ corresponds to the flow region; and we denote it as (S). The problem (see, for instance [3]) reduces to a search for a function $\chi(w)$, analytic in the open rectangle (S), continuous in the closed rectangle and satisfying the boundary conditions

$$\frac{\partial\theta}{\partial\psi} - \theta = e^{-3\tau}\sin\theta - \theta \quad \text{for } \psi = 1, \qquad \theta = 0 \quad \text{for } \psi = 0,$$
$$\theta = 0 \quad \text{for } \varphi = \pm\frac{\pi}{\lambda}$$

Having solved the problem which was posed we can express the relationship between z and w in quadratures. Actually, from (1.9), it follows that

$$dz = \nu^{-\frac{1}{3}} e^{i\chi(w)} dw \tag{1.11}$$

We integrate, bearing in mind that $x = \pi/\lambda$, $y = 0$ for $\phi = \pi/\lambda, \psi = 0$, and obtain

$$z = \frac{\pi}{\lambda} + \nu^{-\frac{1}{3}} \int_{\pi/\lambda}^{w} e^{i\chi(t)} dt \tag{1.12}$$

Because $z = -\pi/\lambda$ when $w = -\pi/\lambda$, the following supplementary condition should be fulfilled:

$$\frac{\pi}{\lambda} = \frac{1}{2}\nu^{-\frac{1}{3}} \int_0^{\lambda/\pi} e^{\tau(t)} \cos\theta(t)\, dt \tag{1.13}$$

2. Green's function for the linear problem. Let us consider the following boundary-value problem for harmonic functions: find function $\theta(x, y)$, harmonic in the open rectangle (S), continuous in the closed rectangle and satisfying the boundary conditions

$$\theta_\psi - \theta = f(\varphi) \quad \text{for } \psi = 1 \tag{2.1}$$
$$\theta = 0 \quad \text{for } \psi = 0, \qquad \theta = 0 \quad \text{for } \varphi = \pm\frac{\pi}{\lambda} \tag{2.2}$$

where $f(\phi)$ is an odd periodic function with period $2\pi/\lambda$. A solution of this problem is given in Littman's article [4].

$$\theta = \int_0^{\pi/\lambda} G(\varphi, \psi, \varphi') f(\varphi')\, d\varphi', \qquad G = \frac{2\lambda}{\pi}\sum_{n=1}^{\infty} \frac{\sinh n\lambda\psi \sin n\lambda\varphi \sin n\lambda\varphi'}{n\lambda\cosh n\lambda - \sinh n\lambda} \tag{2.3}$$

where $G(\phi, \psi, \phi')$ is a Green's function.

If in Formula (2.3) we replace $f(\phi)$ by $e^{-3\tau} \sin\theta - \theta$, we find the conjugate function $\tau(\phi)$ and put $\psi = 1$, the problem posed at the end of Section 1 reduces to nonlinear integral equations. Littman has shown that these equations have solutions which decay in a plane-parallel flow when $\lambda \equiv 0$. Littman's result does not include the more interesting group of solutions which decay in a solitary wave. This is because an analysis of the properties of Green's functions (2.3) is very much more difficult when $\lambda \to 0$ because in such cases the Fourier series degenerates into a Fourier integral. Below we give a transformation of Green's function into a more convenient form for $\psi = 1$.

Let λ_k be the roots of the equation

$$t \cos t - \sin t = 0 \tag{2.4}$$

Theorem 2.1. Function $G(\phi, 1, \phi')$ can be expressed as

$$G = \begin{cases} 3\left[\varphi - \frac{\lambda}{\pi}\varphi\varphi'\right] - \sum\limits_{k=1}^{\infty} \frac{2\sinh(\varphi' - \pi/\lambda)\,\lambda_k \sinh\lambda_k\varphi}{\lambda_k \sinh(\pi\lambda_k/\lambda)} & \text{for } (\varphi < \varphi') \\ 3\left[\varphi' - \frac{\lambda}{\pi}\varphi\varphi'\right] - \sum\limits_{k=1}^{\infty} \frac{2\sinh\lambda\ \varphi' \sinh(\varphi - \pi/\lambda)\,\lambda_k}{\lambda_k \sinh(\pi\lambda_k/\lambda)} & \text{for } (\varphi > \varphi') \end{cases} \tag{2.5}$$

Proof. Note first of all that the following expansion in elementary fractions is valid:

$$\frac{\sinh z}{z\cosh z - \sinh z} = \frac{3}{z^2} + 2\sum_{k=1}^{\infty} \frac{1}{z^2 + \lambda_k^2} \tag{2.6}$$

Replace z by $n\lambda$ to obtain

$$\frac{\sinh n\lambda}{n\lambda\cosh n\lambda - \sinh n\lambda} = \frac{3}{\lambda^2}\frac{1}{n^2} + \frac{2}{\lambda^2}\sum_{k=1}^{\infty}\frac{1}{n^2 + (\lambda_k/\lambda)^2} \tag{2.7}$$

If we insert the expansion into (2.3) and change the order of summation (the validity of this can be proved), we obtain

$$G = \frac{3}{\pi\lambda}\sum_{n=1}^{\infty}\frac{\cos n\lambda(\varphi - \varphi') - \cos n\lambda(\varphi + \varphi')}{n^2} + \\ + \frac{2}{\pi}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\lambda_k/\lambda)\left[\cos n\lambda(\varphi - \varphi') + \cos n\lambda(\varphi + \varphi')\right]}{\lambda_k\left[n^2 + (\lambda_k/\lambda)^2\right]} \tag{2.8}$$

We now make use of the following trigonometrical expansions:

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{1}{4} x^2 - \frac{\pi}{2} |x| + \frac{\pi^2}{6} \tag{2.9}$$

$$(-\pi \leqslant x \leqslant \pi)$$

$$\frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n a}{n^2 + a^2} \cos nx = \frac{\pi}{2} \frac{\cosh ax}{\sinh a\pi} \tag{2.10}$$

Replace x by $x + \pi$ and $x - \pi$ in (2.10), to obtain

$$\frac{\pi}{2} \frac{\cosh a(x-\pi)}{\sinh a\pi} = \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{a}{n^2+a^2} \cos nx \qquad (0 \leqslant x \leqslant 2\pi) \tag{2.11}$$

$$\frac{\pi}{2} \frac{\cosh a(x+\pi)}{\sinh a\pi} = \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{a}{n^2+a^2} \cos nx \qquad (-2\pi \leqslant x \leqslant 0) \tag{2.12}$$

Inserting into Equations (2.9), (2.11) and (2.12) $x = \lambda\ (\phi \pm \phi')$, $a = \lambda_k/\lambda$, we then have

$$\frac{3}{\pi\lambda} \sum_{n=1}^{\infty} \frac{\cos n\lambda(\varphi - \varphi') - \cos n\lambda(\varphi + \varphi')}{n^2} = 3\left[\min(\varphi,\ \varphi') - \frac{\lambda}{\pi} \varphi\varphi'\right] \tag{2.13}$$

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(\lambda_k / \lambda)[\cos n\lambda(\varphi - \varphi') - \cos n\lambda(\varphi + \varphi')]}{n^2 + (\lambda_k / \lambda)^2} =$$

$$= \begin{cases} \dfrac{\cosh(\varphi' - \varphi - \pi/\lambda)\lambda_k - \cosh(\varphi' + \varphi - \pi/\lambda)\lambda_k}{\sinh(\lambda_k \pi / \lambda)} & (\varphi < \varphi') \\ \dfrac{\cosh(\varphi' - \varphi - \pi/\lambda)\lambda_k - \cosh(\varphi' + \varphi - \pi/\lambda)\lambda_k}{\sinh(\lambda_k \pi / \lambda)} & (\varphi > \varphi') \end{cases} \tag{2.14}$$

If we substitute Equations (2.13) and (2.14) in (2.8) we arrive at (2.5). Theorem 2.1 is thus proved.

Theorem 2.2. The solution of the linear problem (2.1) can be represented as follows for $\psi = 1$:

$$\theta = -3S^2 f + 3[1 - (\lambda/\pi)\varphi] Lf + Af \tag{2.15}$$

where L is a functional, S and A are linear operators

$$Lf = \int_0^{\pi/\lambda} \varphi' f(\varphi')\, d\varphi', \qquad Sf = \int_{\pi/\lambda}^{\varphi} f(\varphi')\, d\varphi'$$

$$Af = -2 \sum_{k=1}^{\infty} \frac{1}{\lambda_k \sinh(\pi\lambda_k / \lambda)} \left[\sinh\left(\varphi - \frac{\pi}{\lambda}\right)\lambda_k \int_0^{\varphi} \sinh \lambda_k \varphi' f(\varphi')\, d\varphi' + \right. \tag{2.16}$$

$$+\sinh\lambda_k\varphi\int_{\varphi}^{\pi/\lambda}\sinh\lambda_k\left(\varphi'-\frac{\pi}{\lambda}\right)f(\varphi')\,d\varphi'\Big]$$

Proof. In accordance with Formula (2.3), for $\psi=1$

$$\theta=\int_0^{\pi/\lambda}G(\varphi,\ 1,\ \varphi')f(\varphi')\,d\varphi'=\int_0^{\varphi}G(\varphi,\ 1,\ \varphi')f(\varphi')\,d\varphi'+\int_{\varphi}^{\pi/\lambda}G(\varphi,\ 1,\ \varphi')\,d\varphi' \quad (2.17)$$

If we make use of Equation (2.5) for $G(\phi,\ 1,\ \phi')$ we obtain

$$\theta=3\int_0^{\varphi}\varphi'f(\varphi')\,d\varphi'-\frac{3\lambda}{\pi}\varphi\int_0^{\pi/\lambda}\varphi'f(\varphi')\,d\varphi'+3\varphi\int_{\varphi}^{\pi/\lambda}f(\varphi')\,d\varphi'+\mathrm{A}f \quad (2.18)$$

$$\theta=3\left[1-\frac{\lambda}{\pi}\varphi\right]\mathrm{L}f-3\varphi\int_{\pi/\lambda}^{\varphi}f(\varphi')d\varphi'+3\int_{\pi/\lambda}^{\varphi}\varphi'f(\varphi')\,d\varphi'+\mathrm{A}f \quad (2.19)$$

If we now notice that

$$\mathrm{S}^2f=\varphi\int_{\pi/\lambda}^{\varphi}f(\varphi')\,d\varphi'-\int_{\pi/\lambda}^{\varphi}\varphi'f(\varphi')\,d\varphi' \quad (2.20)$$

Theorem 2.2 is proved.

Theorem 2.3. Function τ, conjugate with θ when $\psi=1$, can be represented as

$$\tau=3\mathrm{S}^3f+\tfrac{3}{2}\frac{\pi}{\lambda}\left(1-\frac{\lambda}{\pi}\varphi\right)^2\mathrm{L}f-\tfrac{3}{2}\mathrm{S}f+\mathrm{B}f+\tau\left(\frac{\pi}{\lambda}\right) \quad (2.21)$$

where B is a linear operator

$$\mathrm{B}f=2\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2\sinh(\pi\lambda_k/\lambda)}\Big[\cosh\left(\varphi-\frac{\pi}{\lambda}\right)\lambda_k\int_0^{\varphi}\sinh\lambda_k\varphi'f(\varphi')\,d\varphi'+$$

$$+\cosh\lambda_k\varphi\int_{\varphi}^{\pi/\lambda}\sinh\left(\varphi'-\frac{\pi}{\lambda}\right)\lambda_kf(\varphi')\,d\varphi'-\int_0^{\pi/\lambda}\sinh\lambda_k\varphi'f(\varphi')\,d\varphi'\Big] \quad (2.22)$$

whilst $\tau(\pi/\lambda)$ is an arbitrary constant.

Proof: Functions $\tau(\phi)/\theta(\phi)$ are connected by Cauchy-Riemann conditions; boundary condition (2.1), therefore, can be written as

$$\frac{d\tau}{d\varphi}=-\theta-f(\varphi)$$

If, instead of θ we substitute its expression (2.15) and integrate between π/λ and ϕ, we arrive at (2.21).

3. Integral expressions of the problem. First approximation. In order to reduce the boundary-value problem, posed in Section 1, to nonlinear integral equations, the following substitution must be made in Formulas (2.15) and (2.21):

$$f(\varphi) = e^{-3\tau} \sin\theta - \theta \tag{3.1}$$

It is well known that an important step in the study of long waves is that of "elongating" or "extending" the independent variable. In a completely formal manner we select some parameter ϵ and subject this to the extension. The physical significance of parameter ϵ will be explained later. Let us assume

$$\begin{gathered}\varphi^\circ = \varepsilon\varphi, \quad \frac{\pi}{\lambda} = \frac{1}{\varepsilon}K, \quad \theta = \varepsilon^3\theta^\circ, \quad \tau = \varepsilon^2\tau^\circ, \\ f^\circ = \frac{1}{\varepsilon^5}\left[e^{-3\varepsilon^2\tau^\circ} \sin \varepsilon^3\theta^\circ - \varepsilon^3\theta^\circ\right]\end{gathered} \tag{3.2}$$

It is evident that

$$\begin{gathered}Sf = \int_{K/\varepsilon}^{\varphi/\varepsilon} f(\varphi')\,d\varphi' = \frac{1}{\varepsilon}\int_K^{\varphi^\circ} f(\varepsilon\varphi^{\circ\prime})\,d\varphi^{\circ\prime} = \varepsilon^4 S^\circ f^\circ \\ S^2 f = \varepsilon^3 S^{\circ 2} f^\circ, \qquad S^3 f = \varepsilon^2 S^{\circ 3} f^\circ\end{gathered} \tag{3.3}$$

Substitute (3.3) in (2.15) and (2.21), and we obtain a system of two integral equations

$$\begin{gathered}\theta^\circ = -3S^{\circ 2}f^\circ + 3\left(1 - \frac{\varphi^\circ}{K}\right)L^\circ f^\circ + 2\pi\varepsilon T f^\circ \\ \tau^\circ = \tau^\circ(K) + 3S^{\circ 3}f^\circ + 3K\left(\frac{\varphi^\circ}{K} - 1\right)^2 L^\circ f^\circ - 2\pi\varepsilon^2 V f^\circ - \frac{3}{2}\varepsilon^2 S^\circ f^\circ\end{gathered} \tag{3.4}$$

where

$$L^\circ f^\circ = \int_0^K \varphi^{\circ\prime} f(\varphi^{\circ\prime})\,d\varphi^{\circ\prime}$$

$$\begin{gathered}Tf^\circ = -\sum_{k=1}^{\infty} \frac{1}{\lambda_k \sinh(\lambda_k K/\varepsilon)}\left[\sinh\frac{\lambda_k}{\varepsilon}(\varphi^\circ - K)\int_0^{\varphi^\circ} f^\circ(\varphi^{\circ\prime}) \sinh\frac{\lambda_k}{\varepsilon}\varphi^{\circ\prime}\,d\varphi^{\circ\prime} + \right. \\ \left. + \sinh\frac{\lambda_k}{\varepsilon}\varphi^\circ \int_{\varphi^\circ}^K f^\circ(\varphi^{\circ\prime}) \sinh\frac{\lambda_k}{\varepsilon}(\varphi^{\circ\prime} - K)\,d\varphi^{\circ\prime}\right]\end{gathered} \tag{3.5}$$

$$Vf^\circ = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2 \sinh(\lambda_k K / \varepsilon)} \left[\cosh\frac{\lambda_k}{\varepsilon}(\varphi^\circ - K) \int_0^{\varphi^\circ} f^\circ(\varphi^{\circ\prime}) \sinh\frac{\lambda_k}{\varepsilon}\varphi^{\circ\prime}\, d\varphi^{\circ\prime} + \right. \tag{3.6}$$

$$\left. + \cosh\frac{\lambda_k}{\varepsilon}\varphi^\circ \int_{\varphi^\circ}^{K} f^\circ(\varphi^{\circ\prime}) \sinh\frac{\lambda_k}{\varepsilon}(\varphi^{\circ\prime} - K)\, d\varphi^{\circ\prime} - \int_0^K \sinh\frac{\lambda_k}{\varepsilon}\varphi^{\circ\prime} f^\circ(\varphi^{\circ\prime})\, d\varphi^{\circ\prime}\right]$$

For simplicity, the sign "0" will be omitted. in what follows.

In first approximation we put $\epsilon = 0$, then Equations (3.4) yield

$$\theta_0 = -3S^2 f_0 + 3\left(1 - \frac{\varphi}{K}\right)\int_0^K \varphi' f_0(\varphi')\, d\varphi' \tag{3.7}$$

$$\tau_0 = \tau_0(K) + 3S^3 f_0 + \tfrac{3}{2} K \left(1 - \frac{\varphi}{K}\right)^2 \int_0^K \varphi' f_0(\varphi')\, d\varphi', \qquad f_0 = -3\tau_0\theta_0 \tag{3.8}$$

It is easy to see that $\theta_0 = -\tau_0'$. If we differentiate the first equation (3.7) twice, we obtain

$$\tau_0''' = 9\tau_0\tau_0' \tag{3.9}$$

The solution of this ordinary differential equation is expressed in elliptical Jacobians (see [4])

$$\tau_0 = \tfrac{1}{3} a^2 \left[2k^2 - 1 - 3k^2 \mathrm{cn}^2\left(a\frac{\sqrt{3}}{2}\varphi\right)\right] \tag{3.10}$$

In these expressions a and k are arbitrary constants, k is the modulus of the elliptical function, a can be chosen arbitrarily (t only affects the relation between ϵ and the physical parameters which determine the flow).

For simplicity we assume that $a = 2/\sqrt{3}$ and we then have

$$\tau_0 = \tfrac{4}{9}(1 - 2k'^2) - \tfrac{4}{3} k^2 \mathrm{cn}^2\varphi, \qquad k'^2 = 1 - k^2 \tag{3.11}$$

Note also that the period K will be a complete elliptic integral of the first order in k. Below we will demonstrate that the system of Equation (3.4) admits a solution which depends on the two parameters ϵ and k'.

4. Variational equations. Let us set

$$\theta = \theta_0 + \delta\theta, \qquad \tau = \tau_0 + \delta\tau \tag{4.1}$$

Notice that Expression (3.1) for f can be represented as

$$f = -3\tau\theta + \varepsilon^2 f_1 \tag{4.2}$$

when f_1 is a complete function of θ, τ and ϵ. We find δf:

$$\delta f = -3\,(\tau_0\delta\theta + \theta_0\delta\tau) + \varepsilon^2\delta f_1 = 3\,(\tau_0{}'\delta\tau - \tau_0\delta\theta) + \varepsilon^2\delta f_1 \tag{4.3}$$

We vary Equation (3.4) and we find

$$\delta\theta = -3S^2\delta f + 3\left(1 - \frac{\varphi}{K}\right)L\delta f + 2\pi\varepsilon T\delta f \tag{4.4}$$

$$\delta\tau = \delta\tau\,(K) + 3S^3\delta f + \frac{3}{2}K\left(1 - \frac{\varphi}{K}\right)^2 L\delta f - 2\pi\varepsilon^2 V\delta f - \frac{3}{2}\varepsilon^2 S\delta f \tag{4.5}$$

If we separate the linear parts of the equations (4.4) and (4.5) and then invert the linear operator in this manner, the problem will reduce to a system of nonlinear integral equations for $\delta\theta$ and $\delta\tau$, which can be solved by the method of successive approximations for small values of ϵ.

Introduce variable y as

$$y = \delta\tau + 2\pi\varepsilon^2 V\delta f + \frac{3}{2}\varepsilon^2 S\,\delta f \tag{4.6}$$

It then follows from (4.4) that

$$-y' = \delta\theta - 2\pi\varepsilon T\delta f \tag{4.7}$$

If we insert this expression into (4.4) and differentiate, we obtain

$$y'' = 3S\delta f - \frac{3}{K}L\,\delta f \tag{4.8}$$

In this expression

$$\delta f = 3(\tau_0 y)' + \delta\sigma,\ \delta\sigma = \varepsilon^2\delta f_1 - 6\pi\varepsilon\,[\varepsilon\tau_0{}'\,V\delta f - \tau_0 T\delta f] - \frac{9}{2}\,\varepsilon^2\tau_0{}'\,S\delta f \tag{4.9}$$

Equation (4.8) can be rewritten as

$$y'' = 9S\,(\tau_0 y)' + \frac{9}{K}L\,(\tau_0 y)' + 3\left(S\,\delta\sigma + \frac{1}{K}L\,\delta\sigma\right) \tag{4.10}$$

However

$$S\,(\tau_0 y)' = \tau_0 y - (\tau_0 y)_{\varphi=K} \tag{4.11}$$

We insert these expressions into Equation (4.10) and obtain

$$y'' - 9\tau_0 y = \frac{c}{K} + u(\varphi), u(\varphi) = 3\left(\mathrm{S}\,\delta\sigma + \frac{1}{K}\mathrm{L}\delta\sigma\right), \quad c = -9\int_0^K \tau_0 y\, d\varphi \quad (4.12)$$

It is easy to prove that the number c can be chosen arbitrarily, i.e. the third of the Equations (4.12) does not impose any limitations on c. In fact, if we integrate Equation (4.12) from 0 to K we arrive at

$$y'(0) - y'(K) - 9\int_0^K \tau_0 y\, d\varphi = c + j \qquad \left(j = \int_0^K u(\varphi) d\varphi\right) \quad (4.13)$$

However, because of (4.7) $y'(0) = y'(K) = 0$, and because of the second of the equations (4.12), $j = 0$, it follows that the third of the expressions (4.12) is fulfilled for any value of c. If we set

$$c = -3\,\mathrm{L}\delta\sigma \quad (4.14)$$

it can only reflect on the way in which ϵ depends on the physical flow parameters. Equation (4.13) will then take the following form

$$y'' - 9\tau_0 y = 3\mathrm{S}\delta\sigma \quad (4.15)$$

and thus the problem of inverting the linear operator reduces to solving an ordinary differential equation.

5. Solution of the differential equation. We are dealing with the equation

$$y'' - 9\tau_0 y = f(\varphi) \quad (5.1)$$

where $f(\phi)$ is an even periodic function with period $2K$. The problem is to find a solution which must also be an even periodic function. We find, first of all, linearly independent solutions of the homogeneous equation. One of the solutions is $z_1(\phi) = r_0'(\phi)$ whilst the second is found from Liouville's formula

$$z_2(\varphi) = z_1(\varphi)\int \frac{d\varphi}{z_1^2(\varphi)} \quad (5.2)$$

From Formula (3.11) we obtain

$$z_1(\varphi) = \mathrm{cn}\varphi\, \mathrm{sn}\varphi\, \mathrm{dn}\,\varphi \quad (5.3)$$

And, it follows that

$$z_2(\varphi) = \mathrm{cn}\varphi\, \mathrm{sn}\varphi\, \mathrm{dn}\varphi \int \frac{d\varphi}{\mathrm{cn}^2\varphi\, \mathrm{sn}^2\varphi\, \mathrm{dn}^2\varphi} \quad (5.4)$$

It is easy to verify the following identities:

$$\frac{1}{\mathrm{cn}^2\varphi\,\mathrm{sn}^2\varphi\,\mathrm{dn}^2\varphi} = \frac{1}{\mathrm{sn}^2\varphi} + \frac{1 + k^2\,\mathrm{cn}^2\varphi}{\mathrm{cn}^2\varphi\,\mathrm{dn}^2\varphi} = \frac{1}{\mathrm{sn}^2\varphi} + \frac{1}{k'^2}\frac{1}{\mathrm{cn}^2\varphi} - \frac{k^4}{k'^2}\frac{1}{\mathrm{dn}^2\varphi} \tag{5.5}$$

Introduce the definitions

$$\Phi_1 = \int_a^\varphi \frac{dt}{\mathrm{sn}^2 t}, \qquad \Phi_2 = \int_0^\varphi \frac{dt}{\mathrm{cn}^2 t}, \qquad \Phi_3 = \int_0^\varphi \frac{dt}{\mathrm{dn}^2 t} \tag{5.6}$$

Then

$$z_2(\varphi) = \mathrm{cn}\varphi\,\mathrm{dn}\varphi\,\mathrm{sn}\varphi\left\{\Phi_1 + \frac{1}{k'^2}\Phi_2 - \frac{k^4}{k'^2}\Phi_3\right\} \tag{5.7}$$

The integrals in Formulas (5.6) are evaluated in an elementary fashion

$$\Phi_1(\varphi) = \varphi - \frac{\mathrm{cn}\varphi\,\mathrm{dn}\varphi}{\mathrm{sn}\varphi} - \int_0^\varphi \mathrm{dn}^2 t\,dt \tag{5.8}$$

$$\Phi_2(\varphi) = \varphi + \frac{1}{k'^2}\left[\frac{\mathrm{sn}\varphi\,\mathrm{dn}\varphi}{\mathrm{cn}\varphi} - \int_0^\varphi \mathrm{dn}^2 t\,dt\right]$$

$$\Phi_3(\varphi) = \frac{1}{k'^2}\left[\int_0^\varphi \mathrm{dn}^2 t\,dt - k^2\frac{\mathrm{sn}\varphi\,\mathrm{cn}\varphi}{\mathrm{dn}\varphi}\right]$$

Theorem 5.1. Function $z_2(\phi)$ can be put in this form:

$$z_2(\varphi) = B\,\varphi z_1(\varphi) + \zeta(\varphi) \tag{5.9}$$

where B is a number, and $\zeta(\phi)$ is an even function of period K.

Proof: We note first of all that

$$\int_0^\varphi \mathrm{dn}^2 t\,dt = \frac{E(k)}{K(k)}\varphi + \chi(\varphi) \tag{5.10}$$

where $\chi(\phi)$ is a periodic function and $E(k)$ and $K(k)$ are complete elliptic integrals of the second and the first kind. Then it follows from (5.9) that

$$\Phi_1 = \left[1 - \frac{E}{K}\right]\varphi + \chi_1(\varphi), \quad \Phi_2 = \left[1 - \frac{E}{k'^2 K}\right]\varphi + \chi_2(\varphi), \quad \Phi_3 = \frac{1}{k'^2}\frac{E}{K}\varphi + \chi_3(\varphi) \tag{5.11}$$

Here χ_1, χ_2, χ_3 are periodic functions. Substituting these expressions in (5.4) we obtain Formula (5.9), and thus

$$B = -\frac{1}{Kk'^4}\left[(1 + k^4 + k'^4)E - K(k'^4 + k'^2)\right] \tag{5.12}$$

and the theorem is proved. We will now demonstrate how to find periodic solutions to Equation (5.1). The general solution depends on two arbitrary constants. The condition of evenness determines one arbitrary constant. Thus the even solution takes the following form:

$$y = z_2(\varphi)\int_K^{\varphi} f(t)z_1(t)\,dt - z_1(\varphi)\int_0^{\varphi} f(t)\,z_2(t)\,dt + cz_2(\varphi) \tag{5.13}$$

We will determine the arbitary constant c from the periodicity condition. Because of periodicity $y(\phi + 2K) - y(\phi) = 0$. On the other hand

$$y(\varphi + 2K) - y(\varphi) = [z_2(\varphi + 2K) - z_2(\varphi)]\int_K^{\varphi+2K} fz_1\,dt +$$
$$+ z_2(\varphi)\Big[\int_K^{\varphi+2K} fz_1\,dt - \int_K^{\varphi} fz_1\,dt\Big] - z_1(\varphi)\Big[\int_0^{\varphi+2K} fz_2\,dt - \int_0^{\varphi} fz_2\,dt\Big] +$$
$$+ c\,[z_2(\varphi + 2K) - z_2(\varphi)] \tag{5.14}$$

Because of (5.9) the equation holds

$$z_2(\varphi + 2K) - z_2(\varphi) = 2BKz_1(\varphi)$$

Owing to the fact that $f(\phi)$ is an even periodic function, whilst $z_1(\phi)$ is odd, we have

$$\int_K^{\varphi+2K} f(t)\,z_1(t)\,dt - \int_K^{\varphi} f(t)\,z_1(t)\,dt = \int_{-K}^{\varphi} fz_1\,dt - \int_K^{\varphi} fz_1\,dt = \int_{-K}^{K} fz_1\,dt = 0$$
$$\int_0^{\varphi+2K} fz_2\,dt - \int_0^{\varphi} fz_2\,dt = \int_K^{\varphi+2K} fz_2\,dt - \int_K^{\varphi} fz_2\,dt = \int_{-K}^{\varphi} fz_2(t + 2K)\,dt - \tag{5.15}$$
$$- \int_K^{\varphi} fz_2\,dt = \int_{-K}^{\varphi} fz_2\,dt + 2BK\int_{-K}^{\varphi} fz_1\,dt + \int_{\varphi}^{K} fz_2\,dt = \int_{-K}^{K} fz_2\,dt + 2BK\int_K^{\varphi} fz_1\,dt$$

If we substitute these expressions in (5.14) we obtain

$$2BKz_1\int_K^{\varphi} fz_1\,dt - z_1\Big[2\int_0^{K} fz_2\,dt + 2BK\int_{-K}^{\varphi} fz_1\,dt\Big] + 2BKz_1(\varphi)\,c = 0 \tag{5.16}$$

This identity will be satisfied if we put

$$c = \frac{1}{BK}\int_0^{K} fz_2\,dt \tag{5.17}$$

Thus we prove the following theorem:

Theorem 5.2. If $f(\phi)$ is an even periodic function with period $2K$, then the equation $y'' - 9r_0 y = f(\phi)$ has a solution which will also be an even periodic function

$$y = \mathrm{N}f \tag{5.18}$$

where N is a linear operator, determined by an equation of the type

$$\mathrm{N}f = z_2(\varphi)\int_K^{\varphi} fz_1\,dt - z_1(\varphi)\int_0^{\varphi} fz_2\,dt + \frac{z_2(\varphi)}{BK}\int_0^K fz_2\,dt \tag{5.19}$$

6. Functional spaces B_1 and B_2. Let B_1 be a Banach space of continuous odd periodic functions of period $2K$, the norm of an element of which is determined by

$$\|\theta\|_{B_1} = \sup_{0\leqslant\varphi\leqslant K}\left[\frac{|\theta(\varphi)|}{\mathrm{dn}\,\varphi}\right] \tag{6.1}$$

Let B_2 be a space of even continuous periodic functions with norm

$$\|\tau\|_{B_2} = \sup_{0\leqslant\varphi\leqslant K}\left[\frac{|\tau(\varphi)|}{\mathrm{dn}\,\varphi}\right] \tag{6.2}$$

Denote by B a Banach space whose elements consist of pairs $\omega(\theta, \tau)$ where $\theta \in B_1$ and $\tau \in B_2$, whilst the norm is equal to

$$\|\omega\| = \{\|\tau\|^2 + \|\theta\|^2\}^{\frac{1}{2}} \tag{6.3}$$

Note that when $k' = 0$ the spaces we have introduced degenerate into spaces which have been used by Friedrichs and Hyers in their proof of the existence of the solitary wave. In what follows we will always assume that

$$0 \leqslant k'^2 \leqslant \tfrac{1}{2} \tag{6.4}$$

We will now deduce several inequalities for elliptic functions which we will have occasion to use frequently.

Note that when $0 \leqslant \phi \leqslant K$

$$0 \leqslant \mathrm{cn}\,\varphi \leqslant \mathrm{dn}\,\varphi \leqslant 1, \qquad \int_0^{\varphi}\mathrm{dn}^2 t\,dt \leqslant \int_0^K dn^2 t\,dt \leqslant \int_0^K \mathrm{dn}\,t\,dt = \frac{\pi}{2} \tag{6.5}$$

Let $\psi = am\ \phi$, we then have

$$\varphi\,\mathrm{cn}\varphi=\cos\psi\int_0^{\psi}\frac{du}{\sqrt{1-k^2\sin^2 u}}\leqslant\frac{\psi\cos\psi}{\sqrt{1-k^2\sin^2\psi}}\leqslant\psi\leqslant\frac{\pi}{2}$$

$$\int_0^{\varphi}\frac{du}{\mathrm{dn}u}=\int_0^{\psi}\frac{dv}{1-k^2\sin^2 v}\leqslant\int_0^{\psi}\frac{dv}{\cos^2 v}=\frac{\sin\psi}{\cos\psi}=\frac{\mathrm{sn}\,\varphi}{\mathrm{cn}\,\varphi}\tag{6.6}$$

Furthermore, it is easy to see that $k^2\mathrm{cn}^2\phi = \mathrm{dn}^2\phi - k'^2$, integrating this identity from 0 to K, we obtain

$$E-k'^2K=k^2\int_0^K \mathrm{cn}^2\varphi\,d\varphi>\frac{1}{2}\int_0^K \mathrm{cn}^2\varphi\,\mathrm{sn}\varphi\,\mathrm{dn}\varphi\,d\varphi=\frac{1}{6}\tag{6.7}$$

Theorem 6.1. Whatever the value of k' from the interval $[0.1/\sqrt{2}]$, operator N acts both from and to B_2 and is limited, i.e. it should be possible to find a constant c_1 independent of k', such that

$$\|Nf\|\leqslant c_1\|f\|$$

Proof: Because $|f| \leqslant \|f\|\ \mathrm{dn}\,\phi$, then, from (5.19) we have

$$|Nf|\leqslant\|f\|\Big(|z_2(\varphi)|\int_{\varphi}^{K} z_1\mathrm{dn}t\,dt+z_1(\varphi)\int_0^{\varphi}|z_2|\,\mathrm{dn}t\,dt+\frac{|z_2|}{BK}\int_0^K|z_2|\,\mathrm{dn}t\,dt\Big)\tag{6.8}$$

Let us first of all evaluate $|z_2(\phi)|$. On the basis of (5.5) $z_2(\phi)$ can be represented as

$$z_2(\varphi)=\mathrm{cn}\varphi\,\mathrm{sn}\varphi\,\mathrm{dn}\varphi\left(\int_a^{\varphi}\frac{dt}{\mathrm{sn}^2t}+\int_0^{\varphi}\frac{1+k^2\mathrm{cn}^2t}{\mathrm{cn}^2t\,\mathrm{dn}^2t}\,dt\right)\tag{6.9}$$

Because the function $\mathrm{dn}\,\phi$ is a decaying one, we have

$$|z_2(\varphi)|<\mathrm{cn}\varphi\,\mathrm{sn}\varphi\,\mathrm{dn}\varphi\left\{|\Phi_1(\varphi)|+\frac{2}{\mathrm{dn}^2\varphi}|\Phi_2(\varphi)|\right\}\tag{6.10}$$

where Φ_1 and Φ_2 are determined from the formulas (5.8). In consequence of (6.5) and (6.6)

$$\mathrm{cn}\varphi\,\mathrm{sn}\varphi\,\mathrm{dn}\varphi\,|\Phi_1(\varphi)|<\tfrac{1}{2}\pi+1+\tfrac{1}{2}\pi=1+\pi\tag{6.11}$$

Now evaluate Φ_2. First of all observe that

$$\int_0^{\varphi}\mathrm{dn}^2t\,dt=\int_0^{\varphi}\mathrm{dn}t\,(\mathrm{dn}t-k\mathrm{cn}t)\,dt-k\mathrm{sn}\varphi=$$

$$=\int_0^{\varphi}\frac{\mathrm{dn}t\,(\mathrm{dn}^2t-k^2\mathrm{cn}^2t)}{\mathrm{dn}t+k\mathrm{cn}t}\,dt-k\mathrm{sn}\varphi=k'^2\int_0^{\varphi}\frac{\mathrm{dn}t}{\mathrm{dn}t+k\mathrm{cn}t}\,dt-k\mathrm{sn}\varphi\tag{6.12}$$

Further

$$\frac{\operatorname{sn}\varphi\operatorname{dn}\varphi}{\operatorname{cn}\varphi}-k\operatorname{sn}\varphi=\frac{\operatorname{sn}\varphi(\operatorname{dn}\varphi-k\operatorname{cn}\varphi)}{\operatorname{cn}\varphi}=\frac{k'^2\operatorname{sn}\varphi}{\operatorname{cn}\varphi(\operatorname{dn}\varphi+k\operatorname{cn}\varphi)}\tag{6.13}$$

Therefore, because of (5.8)

$$\Phi_2(\varphi)=\varphi+\frac{1}{k'^2}\left[\frac{\operatorname{sn}\varphi\operatorname{dn}\varphi}{\operatorname{cn}\varphi}-\int_0^\varphi\operatorname{dn}^2t\,dt\right]=\varphi+\frac{\operatorname{sn}\varphi}{\operatorname{cn}\varphi(\operatorname{dn}\varphi+k\operatorname{sn}\varphi)}+\int_0^\varphi\frac{\operatorname{dn}t\,dt}{\operatorname{dn}t+k\operatorname{cn}t}\tag{6.14}$$

Therefore

$$|\Phi_2(\varphi)|<2\varphi+\frac{1}{\operatorname{cn}\varphi\operatorname{dn}\varphi}\tag{6.15}$$

If we make use of estimate (6.6), we obtain

$$\operatorname{cn}\varphi\operatorname{sn}\varphi\operatorname{dn}\varphi\,|\Phi_2(\varphi)|<1+\pi\tag{6.16}$$

Inserting inequalities (6.16) and (6.11) into (6.10) we find that

$$|z_2(\varphi)|<\frac{3(1+\pi)}{\operatorname{dn}^2\varphi}\tag{6.17}$$

Insert now estimate (6.17) into (6.8)

$$|Nf|\leqslant\|f\|\,3(1+\pi)\left(\frac{1}{\operatorname{dn}^2\varphi}\int_\varphi^K\operatorname{cn}t\operatorname{sn}t\operatorname{dn}^2t\,dt+\right.$$
$$\left.+\operatorname{cn}\varphi\operatorname{sn}\varphi\operatorname{dn}\varphi\int_0^\varphi\frac{dt}{\operatorname{dn}t}+\frac{3(1+\pi)}{BK\operatorname{dn}^2t}\int_0^K\frac{dt}{\operatorname{dn}t}\right)\tag{6.18}$$

But

$$\int_\varphi^K\operatorname{cn}t\operatorname{sn}t\operatorname{dn}^2t\,dt=\frac{1}{2k^2}\,\frac{\operatorname{dn}^3\varphi-k'^3}{3}<\frac{1}{3}\operatorname{dn}^3\varphi,\qquad\int_0^K\frac{dt}{\operatorname{dn}t}=\int_0^{\pi/2}\frac{du}{1-k^2\sin^2u}=\frac{\pi}{2k'}\tag{6.19}$$

Let us estimate, furthermore, the value of BK. In accordance with (5.12)

$$|BK|=\frac{1}{k'^4}\{(1+k^4+k'^4)E-K(k'^2+k'^4)\}\tag{6.20}$$

But, when $0<2k'^2<1$

$$k^4+k'^4=(k^2+k'^2)^2-2k^2k'^2>1-k^2=k'^2$$

Therefore, on the basis of (6.7)

$$|BK| > \frac{1}{k'^4}(1+k'^2)(E-k'^2K) > \frac{1}{6k'^4} > \frac{1}{6k'\mathrm{dn}^3\varphi} \tag{6.21}$$

If we insert the estimates (6.19), (6.20) and (6.21) into (6.18), and take account of (6.6), we arrive at

$$|\mathrm{N}\,f| \leqslant \|f\|\mathrm{dn}\varphi\; 3\;(1+\pi)\left\{\frac{1}{3}+1+9\pi\;(1+\pi)\right\} \tag{6.22}$$

Owing to the fact that the number inside the brackets is independent of k', we have the confirmation of Theorem 6.1.

Theorem 6.2. Operator $Mf = d(Nf)/d\phi$ is bounded and acts from B_2 to B_1.

The proof is similar to that of Theorem 6.1.

Theorem 6.3. Operator $\mathrm{S}f$ operates from B_2 to B_1 and from B_1 to B_2 and is bounded.

Proof: by definition

$$\mathrm{S}f = \int_K^{\varphi} f(\varphi)\, d\varphi \tag{6.23}$$

Therefore

$$|\mathrm{S}\,f| \leqslant \|f\| \int_{\varphi}^{K} \mathrm{dn}t dt = \|f\|\left(\frac{\pi}{2} - am\varphi\right)$$

But if $0 < y < \pi/2$, we have

$$\sin y \geqslant y\left(1-\frac{y^2}{3!}\right) \geqslant y\left(1-\frac{1}{6}\,\frac{\pi^2}{4}\right) \geqslant \frac{1}{2}\,y \tag{6.24}$$

If we take $y = \pi/2 - am\;\;\phi$, we obtain

$$\pi\,/\,2 - am\varphi < 2\cos am\varphi = 2\mathrm{cn}\varphi < 2\mathrm{dn}\varphi$$

and hence it follows that $\|\mathrm{S}f\| < 2\|f\|$.

Theorem 6.4. Operator T acts from space B_1 to B_2 and is bounded, i.e. $\|\mathrm{T}f\| < \epsilon C_2\|f\|$, moreover C_2 is independent of k' and ϵ.

Proof. Remembering the definition (3.5) of the operator T we have

$$|\mathrm{T}f| \leqslant \|f\|\left\{\sum_{m=1}^{\infty} \frac{1}{\lambda_m \sinh(K\lambda_m/\varepsilon)} \times\right.$$

$$\left.\times\left[\sinh\frac{\lambda_m}{\varepsilon}(K-\varphi)\int_0^{\varphi}\mathrm{dn}\,t\sinh\frac{\lambda_m}{\varepsilon}\,t\,dt + \sinh\frac{\lambda_m}{\varepsilon}\,\varphi\int_{\varphi}^{K}\mathrm{dn}\,t\sinh\frac{\lambda_m}{\varepsilon}(K-t)\,dt\right]\right\} \tag{6.25}$$

We give an estimate for the functions

$$A(\varphi)=\int_0^{\varphi}\operatorname{dn} t \sinh\frac{\lambda_m}{\varepsilon} t\, dt, \qquad B(\varphi)=\int_{\varphi}^{K}\operatorname{dn} t \sinh\frac{\lambda_m}{\varepsilon}(K-t)\, dt \tag{6.26}$$

It is obvious that

$$A(\varphi)<\int_0^{\varphi}\operatorname{dn} t \cosh\frac{\lambda_m}{\varepsilon} t\, dt=\frac{\varepsilon}{\lambda_m}\operatorname{dn}\varphi \sinh\frac{\lambda_m}{\varepsilon}\varphi+$$
$$+\frac{\varepsilon}{\lambda_m}\int_0^{\varphi}2k^2\operatorname{cn}\varphi\operatorname{sn}\varphi \sinh\frac{\lambda_m}{\varepsilon} t\, dt<\frac{\varepsilon}{\lambda_m}\left[\operatorname{dn}\varphi \sinh\frac{\lambda_m}{\varepsilon}\varphi+A\right] \tag{6.27}$$

and from this

$$A(\varphi)<\frac{\varepsilon}{\lambda_m}\frac{\operatorname{dn}\varphi \sinh(\lambda_m/\varepsilon\varphi)}{1-\varepsilon/\lambda_m} \tag{6.28}$$

It is even simpler to estimate $B(\phi)$. Because dn ϕ is a monotonic function we have

$$B(\varphi)<\operatorname{dn}\varphi\int_{\varphi}^{K}\sinh\frac{\lambda_m}{\varepsilon}(K-t)\, dt<\operatorname{dn}\varphi\int_{\psi}^{K}\cosh\frac{\lambda_m}{\varepsilon}(K-t)\, dt=$$
$$=\operatorname{dn}\varphi\frac{\varepsilon}{\lambda_m}\sinh\frac{\lambda_m}{\varepsilon}(K-\varphi)<\frac{\varepsilon}{\lambda_m}\frac{1}{1-\varepsilon/\lambda_m}\operatorname{dn}\varphi \sinh\frac{\lambda_m}{\varepsilon}(K-\varphi) \tag{6.29}$$

If we bear in mind that

$$\sinh\frac{\lambda_m}{\varepsilon}\varphi \sinh\frac{\lambda_m}{\varepsilon}(K-\varphi)\leqslant\sinh\frac{\lambda_m}{\varepsilon}\varphi \cosh\frac{\lambda_m}{\varepsilon}(K-\varphi)\leqslant$$
$$\leqslant\frac{1}{2}\left[\sinh\frac{\lambda_m K}{\varepsilon}+\sinh\frac{\lambda_m}{\varepsilon}(K-\varphi)\right]\leqslant\sinh\frac{\lambda_m K}{\varepsilon} \tag{6.30}$$

we obtain

$$|Tf|\leqslant\operatorname{dn}\varphi\|f\|\sum_{m=1}^{\infty}\frac{2\varepsilon}{\lambda_m{}^2}\frac{1}{1-\varepsilon/\lambda_m} \tag{6.31}$$

Let us evaluate the sum of the series. The quantities λ_m satisfy the inequality

$$m\pi<\lambda_m<m\pi+\pi/2$$

Let $\epsilon<\pi/2$, then $1-\epsilon/\lambda_m>1-1/2=1/2$, and thus

$$\sum_{m=1}^{\infty}\frac{2}{\lambda_m{}^2}\frac{1}{1-\varepsilon/\lambda_m}<4\sum_{m=1}^{\infty}\frac{1}{\lambda_m{}^2}<\frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}=\frac{2}{3}$$

Now from (6.31) it follows that

$$\|Tf\| \leqslant \frac{2}{3}\varepsilon\|f\|$$

In a similar way we can show that operator Vf acts from B_1 to B_2 and is limited.

7. Existence theorem. It follows from the results of Section 5 that Equation (4.15) possesses an even periodic solution,

$$y = 3\mathrm{NS}\delta\sigma, \qquad y' = 3\mathrm{MS}\delta\sigma \tag{7.1}$$

where the operator M is determined within the conditions of Theorem 6.2. We then obtain from (4.7) and (4.6) a system of nonlinear equations for $\delta\theta$ and $\delta\tau$

$$\delta\theta = -3\mathrm{MS}\delta\sigma + 2\pi\varepsilon\mathrm{T}\delta f, \qquad \delta\tau = 3\mathrm{NS}\delta\sigma - 2\pi\varepsilon^2\mathrm{V}\delta f - \frac{3}{2}\varepsilon^2\mathrm{S}\delta f \tag{7.2}$$

where $\delta\sigma$ is determined by (4.9). If $\theta \in B_1$ and $\tau \in B_2$, we obtain

$$\delta f \in B_1, \quad \mathrm{V}\delta f \in B_2, \quad \mathrm{T}\delta f \in B_1, \quad \mathrm{S}\delta f \in B_2, \quad \tau_0' \in B_1, \quad \delta f_1 \in B_1$$

Then it follows from (4.9) that $\delta\sigma \in B_1$. Because of Theorems 6.1 and 6.3

$$\mathrm{MS}\delta\sigma \in B_1, \qquad \mathrm{NS}\delta\sigma \in B_2$$

Equations (7.2) can now be written as

$$\delta\theta = \mathrm{E}_1\delta\omega, \qquad \delta\tau = \mathrm{E}_2\delta\omega \tag{7.3}$$

where $\delta\omega$ is in fact $(\delta\theta, \delta\tau)$ whilst E_1 and E_2 are nonlinear operators acting from B to B_1 and B_2, respectively. Denoting by E the pair of operators

$$\mathrm{E} = \{\mathrm{E}_1, \mathrm{E}_2\} \tag{7.4}$$

the system of equations (7.2) can be written as one functional equation in the space B

$$\delta\omega = \mathrm{E}\delta\omega \tag{7.5}$$

It is easy to show that this equation can be solved by successive approximations. Indeed $\mathrm{E}\delta\omega$ can be put in the form

$$\mathrm{E}\delta\omega = \varepsilon(\mathrm{G}\delta f + \mathrm{G}_1\delta f_1) \tag{7.6}$$

where G and G_1 are linear operators acting from B_1 to B. We must show that there exists in the space B a sphere whose radius is such that the operator E maps this sphere onto its interior, whilst the magnification condition of the mapping is fulfilled, i.e.

$$\|E\delta\omega\| \leqslant \|\delta\omega\|, \qquad \|E\delta\omega_1 - E\delta\omega_2\| \leqslant d\|\delta w_1 - \delta\omega_2\| \tag{7.7}$$

where $d < 1$. Denote by F and F_1 the nonlinear operators which relate $\delta\omega \in B$, δf and δf_1, i.e.

$$F\delta\omega = \delta f, \qquad F_1\delta\omega = \delta f_1 \tag{7.8}$$

The proof that the operator E gives a compressed mapping is equivalent to the following: to show that if $\delta\omega$ varies over a limited set in space B, constants M_1, M_2, M_3 and M_4 of such a kind will be found that

$$F\delta\omega\| \leqslant M_1\|\delta\omega\|, \quad \|F_1\delta\omega\| \leqslant M_2\|\delta\omega\|, \quad \|F_1\delta\omega - F_1\delta\omega_1\| \leqslant M_3\|\delta\omega - \delta\omega_1\|$$
$$\|F\delta\omega - F\delta\omega_1\| \leqslant M_4\|\delta\omega - \delta\omega_1\| \tag{7.9}$$

But inequalities (7.9) are easily proved, because δf and δf_1 are analytic functions of $\delta\theta$ and $\delta\tau$. It follows directly from this that when ϵ is sufficiently small operator E will give a compressed mapping. Thus we have been able to prove the following existence theorem.

It is possible to find a number ϵ_0, such that the boundary-value problem (1.10) will have a solution which depends on two parameters ϵ and k', if $0 \leqslant \epsilon \leqslant \epsilon_0$, $0 \leqslant k' \leqslant 1/\sqrt{2}$.

8. Relating of parameters ϵ and k' to the physical parameters determining the motion. It is known that steady wave motion is determined by two non-dimensional parameters. It follows from the results of Section 1 that ν and π/λ can be taken as such parameters. It follows from (1.8) that the period in the physical plane coincides with that of the complex potential. From Equations (3.2) we have

$$\varphi = \frac{\varphi^\circ}{\varepsilon}, \qquad \tau(\varphi) = \varepsilon^2\tau^\circ(\varphi^\circ), \qquad \theta(\varphi) = \varepsilon^3\theta^\circ(\varphi^\circ) \tag{8.1}$$

Because period $\tau^\circ(\phi^\circ)$ and $\theta^\circ(\phi^\circ)$ equals $2K$, period $\tau(\phi)$ equals $2k/\epsilon$, and thus

$$\frac{\pi}{\lambda} = \frac{K(k)}{\varepsilon} \tag{8.2}$$

Now observe that the period can tend to infinity in two cases; (1) k is fixed, $\epsilon \to 0$. From (8.1) it follows that within the limits $\tau(\phi)$ and $\theta(\phi)$ give zero identically, i.e. the cnoidal wave degenerates into a plane parallel flow. This is in fact the case studied by Littman.

(2) ϵ is fixed, $k \to 1$. Within the limits a solitary wave is obtained.

Relation (8.2) gives the relation between λ and ϵ and k'. Expression (1.13) gives the relation between ν and ϵ and k'. It can be written as

$$\frac{\pi}{\lambda} = \nu^{-\frac{1}{3}} \left[\frac{\pi}{\lambda} + \frac{1}{\varepsilon} \int_0^K \{ e^{-\varepsilon^2 \tau^\circ(\varphi^\circ)} \cos \varepsilon^3 \theta^\circ (\varphi^\circ) - 1 \} \, d\varphi^\circ \right] \tag{8.3}$$

or, making use of (8.2) we obtain

$$1 - \nu^{\frac{1}{3}} = \frac{1}{K} \int_0^K [1 - e^{-\varepsilon^2 \tau^\circ(\varphi^\circ)} \cos \varepsilon^3 \theta^\circ (\varphi^\circ)] \, d\varphi^\circ \tag{8.4}$$

If we make use of the theorem of implicit functions, it is easy to demonstrate that for small values of ϵ Equation (8.4) can be solved in terms of ϵ. When ϵ is small ν is close to unity. It is easy to deduce approximate formulas connecting wavelength with amplitude and velocity. This is not done here because these formulas were derived in a simpler manner in [6].

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